Analysis of Boolean Equation Systems through Structure Graphs

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Model Checking

μ-Calculus model checking problem: answer $L \models f$

- L is a *Labelled Transition System*;
- f is a μ-calculus formula;

- Model checking problem $=$ solving Boolean equation systems
- A BES is a sequence of fixed point equations
- Size BES $\mathcal{E}$ encoding $L \models f$ . . . . $\mathcal{O}(|L| \cdot |f|)$
- Solving BES $=$ finding the winner in Parity Games
- Algorithm solving BES $=$ algorithm for computing winner PG
- Computing winner in PG is in UP $\cap$ co-UP . . . . . . [Jurdziński'98]

Observations in practice

Minimisation pays .... [Keiren & Willemse'09]

- Bisimulation minimisations of underlying dependency graph;
- reductions for closed BESs in standard recursive form;
- Solving $\mathcal{E}$ greatly exceeds minimising $\mathcal{E}$ + solving minimised $\mathcal{E}$;

- Solving BES/Computing winners in PG:
  - Small Prog. Meas. [Jurdziński'00] .... $\mathcal{O}(\mathsf{ah}(\mathcal{E}) \cdot m \cdot (\frac{n}{2})^{\frac{\mathsf{ah}(\mathcal{E})}{2}})$
  - Bigstep [Schewe'07] .... $\mathcal{O}(n \cdot m^{\frac{\mathsf{ah}(\mathcal{E})}{3}})$
  - Strat. Impr. [Jurdziński & Vöge'00] .... $\mathcal{O}(2^{\mathsf{ah}(\mathcal{E})} \cdot n \cdot m)$
- Minimisation efficiently and substantially reduces n and m

Contributions of this work

- Graph model for BESs through SOS;
- Bisimulation minimisation for all closed BESs (not just SRF);
- Identified impact of transformation to SRF on minimisation;

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Definition (Boolean Equation System (BES))

A BES $\mathcal{E}$ is defined by the following grammar:

$$\begin{array}{ll} \mathcal{E} & ::= \epsilon \mid (\mu X = f)\ \mathcal{E} \mid (\nu X = f)\ \mathcal{E} \\ f, g & ::= c \mid X \mid f \vee g \mid f \wedge g \end{array}$$

- ϵ is the empty BES;
- μ, ν are least, resp., greatest fixed point signs;
- X is a proposition variable from a set $\mathcal{X}$;
- c is a constant from the set $\{\text{false}, \text{true}\}$;

A BES is in SRF if the formulae are given by the following grammar:

$$f \quad ::= X \mid \bigvee F \mid \bigwedge F$$

- where F is a non-empty set of proposition variables

bound/occurring proposition variables

$$\mathsf{bnd}(\epsilon) = \emptyset \qquad \mathsf{bnd}((\sigma X = f)\ \mathcal{E}) = \mathsf{bnd}(\mathcal{E}) \cup \{X\}$$

$$\mathsf{occ}(\epsilon) = \emptyset \qquad \mathsf{occ}((\sigma X = f)\ \mathcal{E}) = \mathsf{occ}(\mathcal{E}) \cup \mathsf{occ}(f)$$

$$\mathsf{occ}(c) = \emptyset \qquad \mathsf{occ}(f \vee g) = \mathsf{occ}(f) \cup \mathsf{occ}(g)$$
$$\mathsf{occ}(X) = \{X\} \qquad \mathsf{occ}(f \wedge g) = \mathsf{occ}(f) \cup \mathsf{occ}(g)$$

- $\mathcal{E}$ is *closed*: $\mathsf{occ}(\mathcal{E}) \subseteq \mathsf{bnd}(\mathcal{E})$
- $X \trianglelefteq Y$: the equation for X precedes Y's in $\mathcal{E}$
- $\mathcal{E}$ is well-formed: $\trianglelefteq$ is acyclic;

Proposition formulae are interpreted in the context of an environment $\eta{:}\mathcal{X} \to \mathbb{B}$, assigning Boolean values to proposition variables;

Definition (Interpretation of proposition formulae)

Let $\eta{:}\mathcal{X} \to \mathbb{B}$ be an environment. Interpretation $[\![f]\!]\eta$ maps a proposition formula f to true or false:

$$[\![c]\!]\eta = c \qquad [\![X]\!]\eta = \eta(X)$$

$$[\![f \vee g]\!]\eta = [\![f]\!]\eta \vee [\![g]\!]\eta \qquad [\![f \wedge g]\!]\eta = [\![f]\!]\eta \wedge [\![g]\!]\eta$$

The solution to a BES is defined in context of environment $\eta{:}\mathcal{X} \to \mathbb{B}$.
Peculiarities:

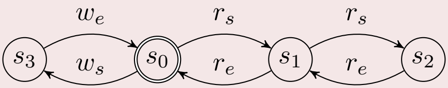
- Solution to closed BES independent of environment:

$$\forall X \in \mathsf{bnd}(\mathcal{E}) : \quad [\![\mathcal{E}]\!]\eta(X) = [\![\mathcal{E}]\!]\eta'(X) \text{ for all } \eta, \eta'$$

- Solution is order-sensitive:

$$[\![(\mu X = Y)\ (\nu Y = X)]\!] \neq [\![(\nu Y = X)\ (\mu X = Y)]\!]$$

Example (Mutual exclusion)



On some path a reader can infinitely often start reading:

$$\nu X.\mu Y.\ \langle r_s \rangle X \vee \langle \overline{r_s} \rangle Y$$

Corresponding BES (translation of [Mader'97]):

$$(\nu X_{s_0} = Y_{s_0})\ (\nu X_{s_1} = Y_{s_1})\ (\nu X_{s_2} = Y_{s_2})\ (\nu X_{s_3} = Y_{s_3})$$
$$(\mu Y_{s_0} = X_{s_1} \vee Y_{s_1})\ (\mu Y_{s_1} = X_{s_2} \vee Y_{s_0})\ (\mu Y_{s_2} = Y_{s_1})\ (\mu Y_{s_3} = Y_{s_0})$$

- rank(X) indicates in which block of like-signed equations X occurs.
- rank(X) is odd iff X is defined in a μ-equation.
- rank(X) inductively defined on structure of BES.

Example (Rank)

$$\begin{array}{llcl} (1) & \mu X & = & (X \wedge Y) \vee Z \\ (2) & \nu Y & = & W \vee (X \wedge Y) \\ (3) & \mu Z & = & Z \\ (3) & \mu W & = & Z \vee (Z \vee W) \end{array}$$

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Aim

Identify a graph structure capturing all information needed for solving equation systems.

It should (at least):

- capture ranks of bound variables;
- reflect the dependencies between bound variables and proposition formulae;
- reflect the dependencies between proposition formulae and its subformulae;
- indicate the top logical operators/constants of proposition formulae;

Definition (Structure Graph)

A structure graph is a graph $\mathcal{G} = \langle T, t, \rightarrow, d \rangle$, where:

- T is a finite set of formulae, and $t \in T$ is the initial formula;
- $\rightarrow \subseteq T \times T$ is a dependency relation;
- $d{:}T \rightarrow (2^{D_\blacktriangle} \cup 2^{D_\blacktriangledown} \cup 2^{D_\top} \cup 2^{D_\bot})$, where, for $e \in \{\blacktriangle, \blacktriangledown, \top, \bot\}$, $D_e = \mathbb{N} \cup \{e\}$, is a term decoration mapping;

$\mathcal{G}_i = \langle T_i, t_i, \rightarrow_i, d_i \rangle$; $R \subseteq T_0 \times T_1$ is a bisimulation if for all $u_0 \; R \; u_1$:

- $d_0(u_0) = d_1(u_1)$;
- for all $v_0 \in T_0$, if $u_0 \rightarrow_0 v_0$, then $u_1 \rightarrow_1 v_1$ for some $v_1 \in T_1$ satisfying $v_0 \; R \; v_1$; (similarly for $u_1 \rightarrow_1 v_1$)

$\mathcal{G}_0 \underline{\leftrightarrow} \mathcal{G}_1$ iff $t_0 \; R \; t_1$ for some bisimulation R

Definition

$\mathcal{G} = \langle T, t, \rightarrow, d \rangle$ is called BESsy if:

- a node t decorated by $\top$ or $\bot$ has no $\rightarrow$ successor;
- a node is decorated by ▲ or ▼ or a rank iff it has a $\rightarrow$ successor;
- a node with multiple $\rightarrow$ successors, is decorated with ▲ or ▼
- a node with rank 0 or 1 is reachable from t,
- the ranks of all reachable nodes form a closed interval

Observation:

- BESsyness is preserved under bisimilarity.

Definition (Transformation BESsy structure graph to BES)

Let $\mathcal{G} = \langle T, t, \rightarrow, d \rangle$ be a BESsy structure graph.

- For each node $u \in T$ with $d(u) \cap \mathbb{N} \neq \emptyset$ introduce an equation:

$$\sigma X_u = rhs(u) \quad \sigma = \begin{cases} \mu & \text{if rank}(X_u) \text{ is odd} \\ \nu & \text{otherwise} \end{cases}$$

$$\text{rank}(X_u) = \min(d(u) \cap \mathbb{N})$$

- The equation system $\mathcal{E}$ is obtained by ordering the equations from left-to-right based on the ranks of the variables;
- The equation system associated to $\mathcal{G}$ is denoted $\mathcal{E}_{\mathcal{G}}$;
- The formula associated to $\mathcal{G}$ is the formula $term(t)$ in the context of the equation system $\mathcal{E}_{\mathcal{G}}$.

For BESsy Structure Graphs, $term$ and rhs are defined as follows:

$$term(u) = \begin{cases} \bigsqcap\{term(u') \mid u \to u'\} & \text{if } d(u) = \{\blacktriangle\}, \\ \bigsqcup\{term(u') \mid u \to u'\} & \text{if } d(u) = \{\blacktriangledown\}, \\ \text{true} & \text{if } \top \in d(u), \\ \text{false} & \text{if } \bot \in d(u), \\ X_u & \text{otherwise,} \end{cases}$$

$$rhs(u) = \begin{cases} \bigsqcap\{term(u') \mid u \to u'\} & \text{if } \blacktriangle \in d(u), \\ \bigsqcup\{term(u') \mid u \to u'\} & \text{if } \blacktriangledown \in d(u), \\ term(u') & \text{otherwise, } u' \text{ s.t. } u \to u'. \end{cases}$$

$\bigsqcap\{t\} = t$ if $t \lessdot \min(T), T \neq \emptyset$, then $\bigsqcap(\{t\} \cup T) = t \wedge \left(\bigsqcap T\right)$

Mapping proposition formulae onto a Structure Graph via SOS

- Flatten nesting hierarchy of the same connective:

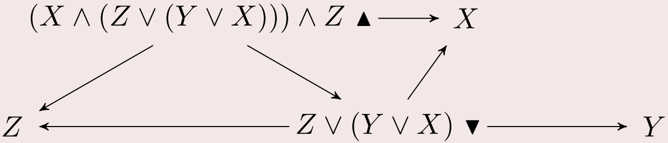
$$\frac{}{\textsf{true}\,\top} \qquad \frac{}{\textsf{false}\ \bot} \qquad \frac{}{(t \wedge t')\blacktriangle} \qquad \frac{}{(t \vee t')\blacktriangledown}$$

$$\frac{t\blacktriangle \quad t \rightarrow u}{t \wedge t' \rightarrow u} \qquad \frac{t\blacktriangledown \quad t \rightarrow u}{t \vee t' \rightarrow u}$$

- Dependencies on variables and constants:

$$\frac{\neg t\blacktriangle}{t \wedge t' \rightarrow t} \qquad \frac{\neg t\blacktriangledown}{t \vee t' \rightarrow t}$$

Example $((X \wedge (Z \vee (Y \vee X))) \wedge Z)$



Mapping equations onto a Structure Graph via SOS

- Ranking the variables:

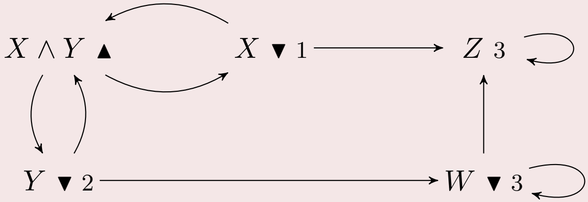
$$\frac{X \in \mathsf{bnd}(\mathcal{E}) \quad \mathsf{rank}(X) = n}{X \pitchfork n}$$

- The structure of a node representing a variable is derived from the right-hand side of the corresponding equation:

$$\frac{\sigma X = t \in \mathcal{E} \quad \neg t\blacktriangle \quad \neg t\blacktriangledown}{X \to t} \qquad \frac{\sigma X = t \in \mathcal{E} \quad t \to u}{X \to u}$$

Example (Equation system and associated structure graph)

$$
\begin{array}{lcl}
\mu X & = & (X \wedge Y) \vee Z \\
\nu Y & = & W \vee (X \wedge Y) \\
\mu Z & = & Z \\
\mu W & = & Z \vee (Z \vee W)
\end{array}
$$



Notation:

- $\mathcal{G}_{\mathcal{E},t}$ denotes structure graph associated to a formula t in the context of an equation system $\mathcal{E}$
- $\mathcal{G}_{\mathcal{E}} = \mathcal{G}_{\mathcal{E},X}$, where $X \in \mathsf{bnd}(\mathcal{E})$ is the least element w.r.t. $\trianglelefteq$

Lemma

Let $\mathcal{E}$ be a non-empty closed equation system. Let t, u, v be proposition formulae such that $\mathsf{occ}(t) \cup \mathsf{occ}(u) \cup \mathsf{occ}(v) \subseteq \mathsf{bnd}(\mathcal{E})$.

$$\mathcal{G}_{\mathcal{E},(t \wedge u) \wedge v} \leftrightarroweq \mathcal{G}_{\mathcal{E},t \wedge (u \wedge v)} \qquad \mathcal{G}_{\mathcal{E},t \wedge u} \leftrightarroweq \mathcal{G}_{\mathcal{E},u \wedge t}$$

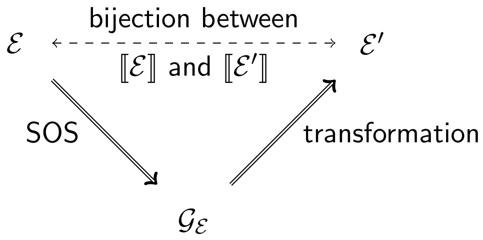
- Idempotency of $\wedge$ is not captured by bisimilarity

Theorem (Preservation of Solution)

- *$\mathcal{E}$ is a non-empty closed BES and $\mathcal{G}_\mathcal{E}$ is its Structure Graph;*
- *$\mathcal{E}'$ is obtained by transforming $\mathcal{G}_\mathcal{E}$ to BES;*

There is a total bijective mapping $h : \mathsf{bnd}(\mathcal{E}) \rightarrow \mathsf{bnd}(\mathcal{E}')$ such that:

$$\forall X \in \mathsf{bnd}(\mathcal{E}) : [\![\mathcal{E}]\!](X) = [\![\mathcal{E}']\!](h(X))$$



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Aim

Bisimilarity preserves and reflects solutions for BESsy structure graphs

Strategy:

$$\begin{array}{ccccc} \mathcal{G}_0 & \xrightarrow{\text{reduce}} & \mathcal{G}_0' & \xrightarrow{\text{normalise}} & \mathcal{G}_0'' \\ \updownarrow\!| & & \updownarrow\!| & & \updownarrow\!| \\ \mathcal{G}_1 & \xrightarrow{\text{reduce}} & \mathcal{G}_1' & \xrightarrow{\text{normalise}} & \mathcal{G}_1'' \end{array}$$

- reduce removes all constant decorations;
- normalise removes all nestings of $\wedge$ and $\vee$;
- normalised structure graph is bisimilar to graph for BES in SRF;

Theorem (Keiren & Willemse'09)

Bisimilarity preserves and reflects solution for BES in SRF

Eliminate false and true nodes

- false node maps to (fresh) node X with $\mu X = X$;
- true node maps to (fresh) node X with $\nu X = X$;
- ranks 0, resp. 1 assigned to new nodes;

Operator reduce is defined by the following deduction rules

$$\frac{t\blacktriangle}{\mathsf{reduce}(t)\blacktriangle} \qquad \frac{t\blacktriangledown}{\mathsf{reduce}(t)\blacktriangledown} \qquad \frac{\neg t\top \quad \neg t\perp \quad t \to u}{\mathsf{reduce}(t) \to \mathsf{reduce}(u)} \qquad \frac{\neg t\top \quad \neg t\perp \quad t \pitchfork n}{\mathsf{reduce}(t) \pitchfork n}$$

$$\frac{t\top}{\mathsf{reduce}(t) \to \mathsf{reduce}(t)} \qquad \frac{t\top}{\mathsf{reduce}(t) \pitchfork 0} \qquad \frac{t\perp}{\mathsf{reduce}(t) \to \mathsf{reduce}(t)} \qquad \frac{t\perp}{\mathsf{reduce}(t) \pitchfork 1}$$

Theorem (Congruence)

Bisimilarity is a congruence for reduce

Removing nesting of $\wedge$ and $\vee$

- Only non-ranked terms occur as subterms in right-hand sides of equations with nested occurrences of $\wedge$ and $\vee$;
- Rank non-ranked nodes based on ranks assigned to successors;

Operator normalise is defined by the following deduction rules

$$\frac{t\blacktriangle}{\mathsf{norm}(t)\blacktriangle} \qquad \frac{t\blacktriangledown}{\mathsf{norm}(t)\blacktriangledown} \qquad \frac{t \longrightarrow u}{\mathsf{norm}(t) \longrightarrow \mathsf{norm}(u)} \qquad \frac{t \pitchfork n}{\mathsf{norm}(t) \pitchfork n}$$

$$\frac{t \not\pitchfork \quad t \longrightarrow u \quad \mathsf{norm}(u) \pitchfork n \quad \forall_v\ t \longrightarrow v \Rightarrow \mathsf{norm}(v) \pitchfork m \wedge m \leq n}{\mathsf{norm}(t) \pitchfork n}$$

Theorem (Congruence)

Bisimilarity is a congruence for normalise

Notation:

- $\mathcal{G}_{\mathsf{norm}(\mathcal{E})}$ denotes the structure graph $\mathcal{G}_{\mathcal{E},\mathsf{norm}(X)}$ where X is the least variable w.r.t. $\trianglelefteq$

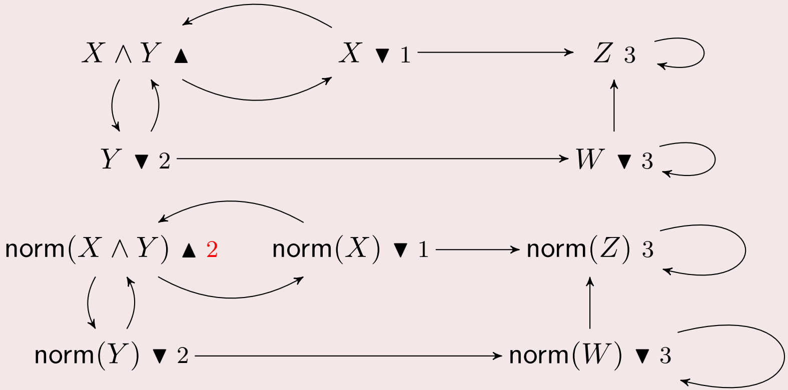
Lemma (Normalisation preserves and reflects solution)

- *$\mathcal{E}$ is a non-empty, closed equation system;*
- *$\mathcal{E}_{norm}$ is the equation system obtained by transforming $\mathcal{G}_{\mathsf{norm}(\mathcal{E})}$ into an equation system;*

There is a total injective mapping $h : \mathsf{bnd}(\mathcal{E}) \to \mathsf{bnd}(\mathcal{E}_{norm})$ such that:

$$\forall X \in \mathsf{bnd}(\mathcal{E}) : [\![\mathcal{E}]\!](X) = [\![\mathcal{E}_{norm}]\!](h(X)).$$

Example (Normalisation)



Theorem

- $\mathcal{E}$ *and* $\mathcal{E}'$ *are non-empty, closed equation systems;*
- f *w.r.t.* $\mathcal{G}_{\mathcal{E}}$ *and* f' *w.r.t.* $\mathcal{G}_{\mathcal{E}'}$ *are bisimilar;*

$$[\![f]\!][\![\mathcal{E}]\!] = [\![f']\!][\![\mathcal{E}']\!]$$

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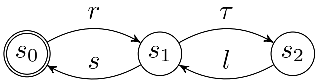
Academic example: LTS modelling an unreliable channel:



- Read a message from the environment . . . . action r;
- Send a message to the environment . . . . action s;
- Lose a message . . . . action l;

Model checking problem

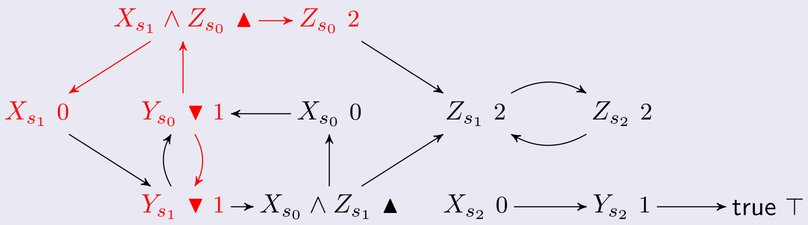
Determine all s_i that can infinitely often never send a message on all reading/sending paths:

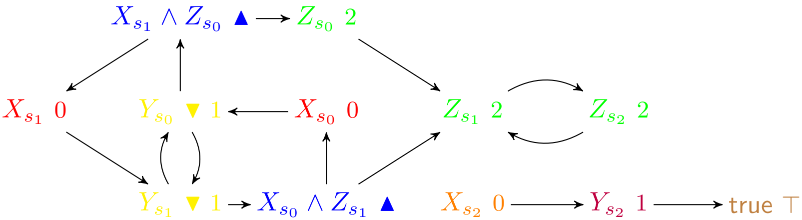
$$s_i \models \nu X.\mu Y.(([r]X \wedge [s]X \wedge (\nu Z.\langle \overline{s} \rangle Z)) \vee ([r]Y \wedge [s]Y))$$

The solution to X_{s_i} (in the BES below) answers whether $s_i \models \phi$

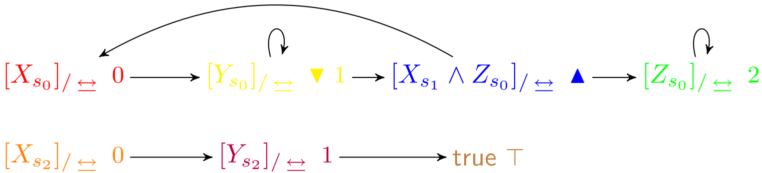
$$(\nu X_{s_0} = Y_{s_0})\ (\nu X_{s_1} = Y_{s_1})\ (\nu X_{s_2} = Y_{s_2})$$
$$(\mu Y_{s_0} = (X_{s_1} \wedge Z_{s_0}) \vee Y_{s_1})\ (\mu Y_{s_1} = (X_{s_0} \wedge Z_{s_1}) \vee Y_{s_0})\ (\mu Y_{s_2} = \mathsf{true})$$
$$(\nu Z_{s_0} = Z_{s_1})\ (\nu Z_{s_1} = Z_{s_2})\ (\nu Z_{s_2} = Z_{s_1})$$

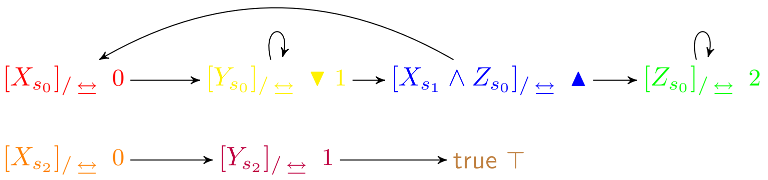
Associated structure graph





Bisimulation minimised structure graph:





Minimised BES

$$
\begin{array}{l}
(\nu X_{[X_{s_0}]_{/\underline{\leftrightarrow}}} = X_{[Y_{s_0}]_{/\underline{\leftrightarrow}}})\ (\nu X_{[X_{s_2}]_{/\underline{\leftrightarrow}}} = X_{[Y_{s_2}]_{/\underline{\leftrightarrow}}})\\
(\mu X_{[Y_{s_0}]_{/\underline{\leftrightarrow}}} = (X_{[X_{s_0}]_{/\underline{\leftrightarrow}}} \wedge X_{[Z_{s_0}]_{/\underline{\leftrightarrow}}}) \vee X_{[Y_{s_0}]_{/\underline{\leftrightarrow}}})\ (\mu X_{[Y_{s_2}]_{/\underline{\leftrightarrow}}} = \mathsf{true})\\
(\nu X_{[Z_{s_0}]_{/\underline{\leftrightarrow}}} = X_{[Z_{s_0}]_{/\underline{\leftrightarrow}}})
\end{array}
$$

Minimised BES:

$$(\nu X_{[X_{s_0}]/\underline{\leftrightarrow}} = X_{[Y_{s_0}]/\underline{\leftrightarrow}})\ (\nu X_{[X_{s_2}]/\underline{\leftrightarrow}} = X_{[Y_{s_2}]/\underline{\leftrightarrow}})$$
$$(\mu X_{[Y_{s_0}]/\underline{\leftrightarrow}} = (X_{[X_{s_0}]/\underline{\leftrightarrow}} \wedge X_{[Z_{s_0}]/\underline{\leftrightarrow}}) \vee X_{[Y_{s_0}]/\underline{\leftrightarrow}})\ (\mu X_{[Y_{s_2}]/\underline{\leftrightarrow}} = \mathsf{true})$$
$$(\nu X_{[Z_{s_0}]/\underline{\leftrightarrow}} = X_{[Z_{s_0}]/\underline{\leftrightarrow}})$$

Observations:

- Answering $s_i \models \phi$ requires solving 5 equations instead of 9. All equations have true as their solution;
- Original LTS: unreducable using strong bisimulation;
- BES size reduction: roughly 50%;

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Future work:

- Extension to open equation systems and concatenation of such equation systems;
- Extension to other coarser notion of equivalence (stuttering equivalence);